

# A Newton conditional gradient method for constrained nonlinear systems

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## Abstract

In this paper, we consider the problem of solving a constrained system of nonlinear equations. We propose an algorithm based on a combination of the Newton and conditional gradient methods, and establish its local convergence analysis. Our analysis is set up by using a majorant condition technique, allowing us to prove in a unified way convergence results for two large families of nonlinear functions. The first one includes functions whose derivative satisfies a Hölder-like condition, and the second one consists of a substantial subclass of analytic functions. Numerical experiments illustrating the applicability of the proposed method are presented, and comparisons with some other methods are discussed.

**Keywords:** constrained nonlinear systems; Newton method; conditional gradient method; local convergence.

## 1 Introduction

In this paper, we consider the problem of finding a solution of the constrained system of nonlinear equations

$$F(x) = 0, \quad x \in C, \quad (1)$$

where  $F : \Omega \rightarrow \mathbb{R}^n$  is a continuously differentiable nonlinear function and  $\Omega \subset \mathbb{R}^n$  is an open set containing the nonempty convex compact set  $C$ . This problem appears in many application areas such as engineering, chemistry and economy. The constraint set may naturally arise in order to exclude solutions of the model with no physical meaning, or it may be considered artificially due to some knowledge about the problem itself (see, for example, [1, 14, 16] and references therein). Different approaches to solve (1) have been proposed in the literature. Many of them are related

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to their unconstrained counterpart having as focus the Newton Methods whenever applicable. Strategies based on different techniques such as trust region, active set and gradient methods have also been used; see, for instance, [1, 2, 14, 15, 16, 20, 22, 25, 26].

In this paper, we propose a Newton conditional gradient (Newton-CondG) method for solving (1), which consists of a Newton step followed by a procedure related to the conditional gradient (CondG) method. The procedure plays the role of getting the Newton iterate back to the constraint set in such a way that the fast convergence of Newton method is maintained. The CondG method (also known as Frank-Wolfe method [4, 9]) requires at each iteration the minimization of a linear functional over the feasible constraint set. This requirement is considered relatively simple and can be fulfilled efficiently for many problems. Moreover, depending on the problem and the structure of the constraint set, linear optimization oracles may provide solutions with specific characteristics leading to important properties such as sparsity and low-rank, see, e.g., [10, 13] for a discussion on this subject. Due to these facts and its simplicity, CondG method have recently received a lot of attention from a theoretical and computational point of view, see for instances [10, 11, 13, 17, 18, 19] and references therein. An interesting approach is to combine variants of CondG method with some superior well designed algorithms; for instance, augmented Lagrangian and accelerated gradient methods, see [17, 18]. In this sense, our combination of CondG and Newton methods seems to be promising.

We present a local convergence analysis of Newton-CondG method. More specifically, we provide an estimate of the convergence radius, for which the well-definedness and the convergence of the method are ensured. Furthermore, results on convergence rates of the method are also established. Our analysis is done via the concept of majorant condition which, besides improving the convergence theory, allows to study Newton type methods in a unified way, see [5, 6, 7, 8] and references therein. Thus, our analysis covers two large families of nonlinear functions, namely, one satisfying a Hölder-like condition, which includes functions with Lipschitz derivative, and another one satisfying a Smale condition, which includes a substantial class of analytic functions. Finally, we also present some numerical experiments illustrating the applicability of our method and discuss its behavior compared with other methods.

This paper is organized as follows. In Section 2, we study a certain scalar sequence generated by a Newton-type method. The Newton-CondG method and its convergence analysis are discussed in Section 3. Section 4 specializes our main convergence result for functions satisfying Hölder-like and Smale conditions. In Section 5, we present some numerical experiments illustrating the applicability of the proposed method.

**Notations and basic assumptions:** Throughout this paper, we assume that  $F : \Omega \rightarrow \mathbb{R}^n$  is a continuously differentiable nonlinear function where  $\Omega \subset \mathbb{R}^n$  is an open set containing nonempty convex compact set  $C$ . The Jacobian matrix of  $F$  at  $x \in \Omega$  is denoted by  $F'(x)$ . We also assume that there exists  $x_* \in C$  such that  $F(x_*) = 0$  and  $F'(x_*)$  is nonsingular. Let the inner product and its associated norm in  $\mathbb{R}^n$  be denoted by  $\langle \cdot, \cdot \rangle$  and  $\| \cdot \|$ , respectively. The open ball centered at  $a \in \mathbb{R}^n$  and radius  $\delta > 0$  is denoted by  $B(a, \delta)$ . For a given linear operator  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , we also

use  $\|\cdot\|$  to denote its norm, which is defined by

$$\|T\| := \sup\{\|Tx\|, \|x\| \leq 1\}.$$

## 2 Preliminaries results

Our goal in this section is to study the behavior of a scalar sequence generated by a Newton-type method applied to solve

$$f(t) = 0,$$

where  $f : [0, R) \rightarrow \mathbb{R}$  is a continuously differentiable function such that

**h1.**  $f(0) = 0$  and  $f'(0) = -1$ ;

**h2.**  $f'$  is strictly increasing.

Although **h1** implies that  $t_* = 0$  is a solution of the above equation, the convergence properties of this scalar sequence will be directly associated to the sequence generated by Newton-CondG method. First, consider the scalar  $\nu$  given by

$$\nu := \sup\{t \in [0, R) : f'(t) < 0\}. \quad (2)$$

Since  $f'$  is continuous and  $f'(0) = -1$ , it follows that  $\nu > 0$ . Moreover, **h2** implies that  $f'(t) < 0$  for all  $t \in [0, \nu)$ . Hence, the following Newton iteration map for  $f$  is well defined:

$$\begin{aligned} n_f : [0, \nu) &\rightarrow \mathbb{R} \\ t &\mapsto t - f(t)/f'(t). \end{aligned} \quad (3)$$

Let us also consider the scalars  $\lambda$  and  $\rho$  such that

$$\lambda \in [0, 1), \quad \rho := \sup \left\{ \delta \in (0, \nu) : (1 + \lambda) \frac{|n_f(t)|}{t} + \lambda < 1, \ t \in (0, \delta) \right\}. \quad (4)$$

We now present some properties of the Newton iteration map  $n_f$  and show that  $\rho > 0$ .

**Proposition 1.** *The following statements hold:*

a)  $n_f(t) < 0$  for all  $t \in (0, \nu)$ ;

b)  $\lim_{t \downarrow 0} |n_f(t)|/t = 0$ ;

c) the scalar  $\rho$  is positive and

$$0 < (1 + \lambda)|n_f(t)| + t\lambda < t, \quad \forall t \in (0, \rho). \quad (5)$$

*Proof.* (a) From condition **h2** we see that  $f'$  is strictly increasing in  $[0, R)$ , in particular,  $f$  is strictly convex. Hence, since  $\nu \leq R$  (see (2)), we obtain  $f(0) > f(t) + f'(t)(0 - t)$ , for any  $t \in (0, \nu)$  which combined with  $f(0) = 0$  and  $f'(t) < 0$  for any  $t \in (0, \nu)$ , proves item (a).

(b) In view of item (a) and the fact that  $f(0) = 0$ , we obtain

$$\frac{|n_f(t)|}{t} = \frac{1}{t} \left( \frac{f(t)}{f'(t)} - t \right) = \frac{1}{f'(t)} \frac{f(t) - f(0)}{t - 0} - 1, \quad \forall t \in (0, \nu). \quad (6)$$

As  $f'(0) \neq 0$ , item (b) follows by taking limit in (6), as  $t \downarrow 0$ .

(c) Since  $0 \leq \lambda < 1$ , using items (a) and (b), we conclude that there exists  $\delta > 0$  such that

$$0 < \frac{|n_f(t)|}{t} < \frac{1 - \lambda}{1 + \lambda}, \quad \forall t \in (0, \delta),$$

or, equivalently,

$$0 < (1 + \lambda) \frac{|n_f(t)|}{t} + \lambda < 1, \quad \forall t \in (0, \delta).$$

Therefore,  $\rho$  is positive and (5) trivially holds.  $\square$

Let  $t_0 \in (0, \rho)$  and  $\{\theta_k\} \subset [0, +\infty)$  be given, and define the scalar sequence  $\{t_k\}$  by

$$t_{k+1} = (1 + \sqrt{2\theta_k})|n_f(t_k)| + \sqrt{2\theta_k}t_k, \quad \forall k \geq 0. \quad (7)$$

**Corollary 2.** Assume that  $\{\theta_k\} \subset [0, \lambda^2/2]$ . Then the sequence  $\{t_k\}$  is well defined, strictly decreasing and converges to 0. Moreover,  $\limsup_{k \rightarrow \infty} t_{k+1}/t_k = \sqrt{2\tilde{\theta}}$ , where  $\tilde{\theta} = \limsup_{k \rightarrow \infty} \theta_k$ .

*Proof.* First of all, since  $(0, \rho) \subset \text{dom}(n_f)$ , in order to show the well definedness of  $\{t_k\}$  is sufficient to prove that  $t_k \in (0, \rho)$  for all  $k$ . Let us prove this latter inclusion by induction on  $k$ . As  $t_0 \in (0, \rho)$ , the statement trivially holds for  $k = 0$ . Now, assume that it holds for some  $k \geq 0$ . Hence,  $t_k \in \text{dom}(n_f)$  and it follows from Proposition 1 (a) and (7) that

$$0 < t_{k+1} = (1 + \sqrt{2\theta_k})|n_f(t_k)| + \sqrt{2\theta_k}t_k \leq (1 + \lambda)|n_f(t_k)| + \lambda t_k < t_k, \quad (8)$$

where the second and third inequalities are due to  $0 \leq \sqrt{2\theta_k} \leq \lambda$  and (5), respectively. It follows from (8) and the induction assumption that  $t_{k+1} \in (0, \rho)$ , concluding the induction proof. Thus,  $\{t_k\}$  is well defined and (8) also implies that it is strictly decreasing. As a consequence, we have  $\{t_k\}$  converges to some  $t_* \in [0, \rho)$ . Thus, since  $n_f(\cdot)$  is continuous, taking limit superior in (8) as  $k \rightarrow \infty$ , we obtain, in particular,

$$t_* \leq (1 + \lambda)|n_f(t_*)| + \lambda t_*.$$

Therefore, (5) implies that  $t_* = 0$ . Now, using (7),  $\lim_{k \rightarrow \infty} t_k = 0$  and Proposition 1 (b), we have

$$\limsup_{k \rightarrow \infty} \frac{t_{k+1}}{t_k} = \limsup_{k \rightarrow \infty} \left( (1 + \sqrt{2\theta_k}) \frac{|n_f(t_k)|}{t_k} + \sqrt{2\theta_k} \right) = \sqrt{2\tilde{\theta}},$$

concluding the proof of the corollary.  $\square$

### 3 The method and its convergence analysis

In this section, we propose a Newton conditional gradient method to solve (1) and discuss its local convergence results.

#### 3.1 The Newton-CondG method

In this subsection, we present a method for solving (1) which consists of a Newton step followed by a procedure related to an inexact conditional gradient method. This procedure is used in order to retrieve the Newton iterate back to the constraint set  $C$  in such a way that the fast convergence of the sequence generated by the method is ensured. We assume the existence of a linear optimization oracle (LO oracle) capable of minimizing linear functions over  $C$ . The Newton conditional gradient method is formally described as follows.

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#### Newton-CondG method

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**Step 0.** Let  $x_0 \in C$  and  $\{\theta_j\} \subset [0, \infty)$  be given and set  $k = 0$ .

**Step 1.** If  $F(x_k) = 0$ , then **stop**; otherwise, compute  $s_k \in \mathbb{R}^n$  and  $y_k \in \mathbb{R}^n$  such that

$$F'(x_k)s_k = -F(x_k), \quad y_k = x_k + s_k. \quad (9)$$

**Step 2.** Given  $\theta_k \geq 0$ , use CondG procedure to obtain  $x_{k+1} \in C$  as

$$x_{k+1} = \text{CondG}(y_k, x_k, \theta_k \|s_k\|^2). \quad (10)$$

**Step 3.** Set  $k \leftarrow k + 1$ , and go to step 1.

**end**

**CondG procedure**  $z = \text{CondG}(y, x, \varepsilon)$

**P0.** Set  $z_1 = x$  and  $t = 1$ .

**P1.** Use the LO oracle to compute an optimal solution  $u_t$  of

$$g_t^* = \min_{u \in C} \{\langle z_t - y, u - z_t \rangle\}.$$

**P2.** If  $g_t^* \geq -\varepsilon$ , set  $z = z_t$  and **stop** the procedure; otherwise, compute  $\alpha_t \in (0, 1]$  and  $z_{t+1}$  as

$$\alpha_t := \min \left\{ 1, \frac{-g_t^*}{\|u_t - z_t\|^2} \right\}, \quad z_{t+1} = z_t + \alpha_t(u_t - z_t).$$

**P3.** Set  $t \leftarrow t + 1$ , and go to **P1**.

**end procedure**

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Some remarks regarding Newton-CondG method are in order. First, in step 1, we check whether the current iterate  $x_k$  is a solution, and if not, we execute a Newton step. Second, since the iterate  $y_k$  obtained after a Newton step may be infeasible to the constraint set  $C$ , we apply an inexact conditional gradient method in order to get the new iterate  $x_{k+1}$  in  $C$ . As mentioned before, this method requires an oracle which is assumed to be able to minimize linear functions over the constraint set. It is clear that this may be done efficiently for a wide class of sets, since many methods for minimizing linear functions are well established in the literature. Third, if CondG procedure computes  $g_t^* \geq -\varepsilon$  then it stops and out put  $z_t \in C$ . Hence, if the procedure continues, we have  $g_t^* < -\varepsilon \leq 0$  which implies that the stepsize  $\alpha_t$  is well defined and belongs to  $(0, 1]$ .

### 3.2 Convergence of Newton-CondG method

In this subsection, we discuss the convergence behavior of Newton-CondG method. First, we present the concept of majorant functions and some of their properties. Then, some properties of CondG procedure are studied. Finally, we state and prove our main result. Basically, our main theorem specifies a convergence radius of the method and analyze its convergence results. Moreover, it also present the relationship between the Newton-CondG sequence  $\{x_k\}$  and sequence  $\{t_k\}$  defined in (7), which will be associated to our majorant function.

We start by defining, for any given  $R \in (0, +\infty]$ , the scalar  $\kappa$  as

$$\kappa := \kappa(\Omega, R) = \sup \{t \in [0, R) : B(x_*, t) \subset \Omega\}. \quad (11)$$

In order to analyze the convergence properties of Newton-CondG method, we consider the concept of majorant functions which has the advantage of presenting, in a unified way, convergence result for different classes of nonlinear functions; more details about the majorant condition can be found in [5, 6, 7, 8].

**Definition 1.** Let  $R \in (0, +\infty]$  be given and consider  $\kappa$  as in (11). A function  $f : [0, R) \rightarrow \mathbb{R}$  continuously differentiable is said to be a majorant function for the function  $F$  on  $B(x_*, \kappa)$  if and only if

$$\|F'(x_*)^{-1} [F'(x) - F'(x_* + \tau(x - x_*))]\| \leq f'(\|x - x_*\|) - f'(\tau\|x - x_*\|), \quad (12)$$

for all  $\tau \in [0, 1]$  and  $x \in B(x_*, \kappa)$ , and conditions **h1** and **h2** are satisfied.

To illustrate Definition 1, let  $\mathcal{L}$  be the class of continuously differentiable functions  $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $G'(x_*)$  is non-singular and the following Hölder type condition is satisfied

$$\|G'(x_*)^{-1}(G'(x) - G'(x_* + \tau(x - x_*)))\| \leq K(1 - \tau^p)\|x - x_*\|^p, \quad x \in \mathbb{R}^n, \quad \tau \in [0, 1],$$

for some  $K > 0$  and  $0 < p \leq 1$ . It is easy to see that the function  $f : [0, \infty) \rightarrow \mathbb{R}$  given by  $f(t) = (K/(p+1))t^{p+1} - t$  is a majorant function for any  $G \in \mathcal{L}$ . It is worth pointing out that the class  $\mathcal{L}$  includes, in particular, functions  $G$  with Lipschitz continuous derivative such that  $G'(x_*)$  is non-singular. Section 4 contains more details on this class of functions and also another one for analytic functions satisfying a Smale condition.

Before presenting some properties of majorant functions, let  $R \in (0, +\infty]$  be given and  $\kappa$  as defined in (11), and consider the following condition:

**A1.** the function  $F$  has a majorant function  $f : [0, R) \rightarrow \mathbb{R}$  on  $B(x_*, \kappa)$ .

Let us now present a result which is fundamental for the convergence analysis of Newton-CondG method. More precisely, it highlights the relationship between the nonlinear function  $F$  and its majorant function  $f$ .

**Lemma 3.** Assume that **A1** holds, and let  $x \in B(x^*, \min\{\kappa, \nu\})$ , where  $\nu$  is defined in (2). Then the function  $F'(x)$  is invertible and the following estimates hold:

- a)  $\|F'(x)^{-1}F'(x_*)\| \leq 1/|f'(\|x - x_*\|)|$ ;
- b)  $\|F'(x)^{-1}F(x)\| \leq f(\|x - x_*\|)/f'(\|x - x_*\|)$ ;
- c)  $\|F'(x_*)^{-1}[F(x_*) - F(x) - F'(x)(x_* - x)]\| \leq f'(\|x - x_*\|)\|x - x_*\| - f(\|x - x_*\|)$ .

*Proof.* The proof follows the same pattern as the proofs of Lemmas 10, 11 and 12 in [6]. □

Next, we present two results which contain some basic properties of CondG procedure.

**Lemma 4.** For any  $y, \tilde{y} \in \mathbb{R}^n$ ,  $x, \tilde{x} \in C$  and  $\mu \geq 0$ , we have

$$\|CondG(y, x, \mu) - CondG(\tilde{y}, \tilde{x}, 0)\| \leq \|y - \tilde{y}\| + \sqrt{2\mu}.$$

*Proof.* Let us denote  $z = CondG(y, x, \mu)$  and  $\tilde{z} = CondG(\tilde{y}, \tilde{x}, 0)$ . It follows from CondG procedure that  $z, \tilde{z} \in C$  and

$$\langle z - y, \tilde{z} - z \rangle \geq -\mu, \quad \langle \tilde{z} - \tilde{y}, z - \tilde{z} \rangle \geq 0. \quad (13)$$

On the other hand, after some simple algebraic manipulations we have

$$\|y - \tilde{y}\|^2 = \|z - \tilde{z}\|^2 + \|(y - z) - (\tilde{y} - \tilde{z})\|^2 + 2\langle y - z - (\tilde{y} - \tilde{z}), z - \tilde{z} \rangle,$$

which implies that

$$\|z - \tilde{z}\|^2 \leq \|y - \tilde{y}\|^2 + 2\langle z - y, z - \tilde{z} \rangle + 2\langle \tilde{y} - \tilde{z}, z - \tilde{z} \rangle.$$

The last inequality together with (13) yields

$$\|z - \tilde{z}\|^2 \leq \|y - \tilde{y}\|^2 + 2\mu,$$

and then

$$\|z - \tilde{z}\| \leq \|y - \tilde{y}\| + \sqrt{2\mu},$$

which combined with the definitions of  $z$  and  $\tilde{z}$  proves the lemma.  $\square$

The following scalar will be used in the convergence analysis of Newton-CondG method:

$$r := \min\{\rho, \kappa\}, \quad (14)$$

where  $\rho$  and  $\kappa$  are defined in (4) and (11), respectively. Since Lemma 3 implies that  $F'(x)$  is invertible for any  $x \in B(x_*, r)$ , we see that the following Newton iteration map is well defined:

$$\begin{aligned} N_F : B(x_*, r) &\rightarrow \mathbb{R}^n \\ x &\mapsto x - S_F(x), \end{aligned} \quad (15)$$

where

$$S_F(x) := F'(x)^{-1}F(x). \quad (16)$$

**Lemma 5.** *Assume that **A1** holds, and let  $x \in C \cap B(x^*, r)$  and  $\theta \geq 0$ . Then there holds*

$$\| \text{CondG}(N_F(x), x, \theta \|S_F(x)\|^2) - x_* \| \leq (1 + \sqrt{2\theta})|n_f(\|x - x_*\|)| + \sqrt{2\theta}\|x - x_*\|, \quad (17)$$

where  $n_f$  is defined in (3). As a consequence, letting  $\lambda$  as in (4), if  $\sqrt{2\theta} \leq \lambda$ , then

$$\text{CondG}(N_F(x), x, \theta \|S_F(x)\|^2) \in C \cap B(x^*, r). \quad (18)$$

*Proof.* It follows from Lemma 4 with  $y = N_F(x)$ ,  $\tilde{y} = N_F(x_*)$ ,  $\tilde{x} = x_*$  and  $\mu = \theta \|S_F(x)\|^2$  that

$$\| \text{CondG}(N_F(x), x, \theta \|S_F(x)\|^2) - \text{CondG}(N_F(x_*), x_*, 0) \| \leq \|N_F(x) - N_F(x_*)\| + \sqrt{2\theta}\|S_F(x)\|.$$

It is easy to see from CondG procedure that  $\text{CondG}(x, x, 0) = x$ , for all  $x \in C$ . Hence, since  $F(x_*) = 0$  implies that  $N_F(x_*) = x_*$ , we have  $\text{CondG}(N_F(x_*), x_*, 0) = x_*$ . Thus, the last inequality gives

$$\| \text{CondG}(N_F(x), x, \theta \|S_F(x)\|^2) - x_* \| \leq \|N_F(x) - x_*\| + \sqrt{2\theta}\|S_F(x)\|. \quad (19)$$

On the other hand, using (15) and  $F(x_*) = 0$ , we have

$$N_F(x) - x_* = F'(x)^{-1} [F(x_*) - F(x) - F'(x)(x_* - x)],$$

which combined with Lemma 3 (a) and (c), and the fact that  $f'(\|x - x_*\|) < 0$  gives

$$\|N_F(x) - x_*\| \leq \frac{f'(\|x - x_*\|)\|x - x_*\| - f(\|x - x_*\|)}{|f'(\|x - x_*\|)|} = \frac{f(\|x - x_*\|)}{f'(\|x - x_*\|)} - \|x - x_*\|.$$



Hence, it follows from (19), (16) and Lemma 3(b) that

$$\begin{aligned}\|\text{CondG}(N_F(x), x, \theta \|S_F(x)\|^2) - x_*\| &\leq (1 + \sqrt{2\theta}) \frac{f(\|x - x_*\|)}{f'(\|x - x_*\|)} - \|x - x_*\| \\ &= (1 + \sqrt{2\theta}) |n_f(\|x - x_*\|)| + \sqrt{2\theta} \|x - x_*\|,\end{aligned}$$

where the equality is due to definition of  $n_f$  in (3) and Proposition 1 (a). Thus, (17) is proved.

Now, since  $\sqrt{2\theta} \leq \lambda$  and  $0 < \|x - x_*\| < r \leq \rho$ , it follows from (5) with  $t = \|x - x_*\|$  that

$$(1 + \sqrt{2\theta}) |n_f(\|x - x_*\|)| + \sqrt{2\theta} \|x - x_*\| < \|x - x_*\| < r,$$

which together with (17) gives  $\text{CondG}(N_F(x), x, \theta \|S_F(x)\|^2) \in B(x^*, r)$ . Therefore, (18) follows from the fact that points generated by CondG procedure belong to  $C$ .  $\square$

In the following, we state and prove our main convergence result.

**Theorem 6.** *Let  $\lambda, \rho$  and  $\kappa$  be as in (4) and (11), and consider*

$$r = \min\{\rho, \kappa\}. \quad (20)$$

*Assume that **A1** holds, and let  $\{\theta_k\}$  and  $x_0$  be given in step 0 of Newton-CondG method. If  $\{\theta_k\} \subset [0, \lambda^2/2]$  and  $x_0 \in C \cap B(x_*, r) \setminus \{x_*\}$ , then Newton-CondG method generates a sequence  $\{x_k\}$  which is contained in  $B(x_*, r) \cap C$ , converges to  $x_*$  and satisfies*

$$\limsup_{k \rightarrow \infty} [\|x_{k+1} - x_*\| / \|x_k - x_*\|] \leq \sqrt{2\tilde{\theta}}, \quad (21)$$

where  $\tilde{\theta} = \limsup_{k \rightarrow \infty} \theta_k$ . Moreover, given  $0 \leq p \leq 1$  and  $n_f$  as in (3), if the following assumption holds

**h3.** *the function  $(0, \nu) \ni t \mapsto |n_f(t)|/t^{p+1}$  is strictly increasing;*

then, for all integer  $k \geq 0$ , we have

$$\|x_{k+1} - x_*\| \leq (1 + \lambda) \frac{|n_f(t_0)|}{t_0^{p+1}} \|x_k - x_*\|^{p+1} + \lambda \|x_k - x_*\|, \quad (22)$$

and

$$\|x_k - x_*\| \leq t_k, \quad (23)$$

where  $\{t_k\}$  is as defined in (7) with  $t_0 = \|x_0 - x_*\|$ .

*Proof.* First of all, it is easy to see from (15), (16) and Newton-CondG method that

$$x_{k+1} = \text{CondG}(N_F(x_k), x_k, \theta_k \|S_F(x_k)\|^2). \quad (24)$$

Hence, since  $x_0 \in C \cap B(x_*, r) \setminus \{x_*\}$ , it follows from the first statement of Lemma 3, inclusion (18) with  $x = x_k$  and  $\theta = \theta_k$ , and a simple induction argument that Newton-CondG method generates a sequence  $\{x_k\}$  contained in  $B(x_*, r) \cap C$ .

We will now prove that  $\{x_k\}$  converges to  $x_*$ . Since for all  $k \geq 0$ ,  $\|x_k - x_*\| < r \leq \rho$ , it follows from (24) and inequality (17) with  $x = x_k$  and  $\theta = \theta_k$  that, for all  $k \geq 0$ ,

$$\begin{aligned} \|x_{k+1} - x_*\| &\leq (1 + \sqrt{2\theta_k})|n_f(\|x_k - x_*\|)| + \sqrt{2\theta_k}\|x_k - x_*\| \\ &\leq (1 + \lambda)|n_f(\|x_k - x_*\|)| + \lambda\|x_k - x_*\| \\ &< \|x_k - x_*\|, \end{aligned} \quad (25)$$

where the second and the third inequalities are due to  $\sqrt{2\theta_k} \leq \lambda$  and (5) with  $t = \|x_k - x_*\|$ , respectively. Hence,  $\{\|x_k - x_*\|\}$  converges to some  $\ell_* \in [0, \rho)$ . Thus, as  $n_f(\cdot)$  is continuous in  $[0, \rho)$ , (25) implies, in particular, that

$$\ell_* \leq (1 + \lambda)|n_f(\ell_*)| + \lambda\ell_*.$$

Therefore, due to (5) we must have  $\ell_* = 0$ , proving that  $x_k \rightarrow x_*$ .

We also see from (25) that, for all  $k \geq 0$ ,

$$\frac{\|x_{k+1} - x_*\|}{\|x_k - x_*\|} \leq (1 + \sqrt{2\theta_k}) \frac{|n_f(\|x_k - x_*\|)|}{\|x_k - x_*\|} + \sqrt{2\theta_k}.$$

The asymptotic rate (21) follows by taking limit superior in the last inequality as  $k \rightarrow \infty$  and using  $\|x_k - x_*\| \rightarrow 0$ , Proposition 1 (b) and  $\limsup_{k \rightarrow \infty} \theta_k = \tilde{\theta}$ .

In order to prove the second part of the theorem, let us assume that **h3** holds. In view of (25), we have  $\|x_k - x_*\| \leq \|x_0 - x_*\| = t_0$  and

$$\|x_{k+1} - x_*\| \leq (1 + \sqrt{2\theta_k}) \frac{|n_f(\|x_k - x_*\|)|}{\|x_k - x_*\|^{p+1}} \|x_k - x_*\|^{p+1} + \sqrt{2\theta_k}\|x_k - x_*\|, \quad \forall k \geq 0. \quad (26)$$

Therefore, (22) follows from assumption **h3** and  $\sqrt{2\theta_k} \leq \lambda$ .

Let us now show inequality (23) by induction. Since  $t_0 = \|x_0 - x_*\|$ , it trivially holds for  $k = 0$ . Assume that  $\|x_k - x_*\| \leq t_k$  for some  $k \geq 0$ . Hence, (26) together with **h3** yields

$$\begin{aligned} \|x_{k+1} - x_*\| &\leq (1 + \sqrt{2\theta_k}) \frac{|n_f(t_k)|}{t_k^{p+1}} \|x_k - x_*\|^{p+1} + \sqrt{2\theta_k}\|x_k - x_*\| \\ &\leq (1 + \sqrt{2\theta_k})|n_f(t_k)| + \sqrt{2\theta_k}t_k = t_{k+1}. \end{aligned}$$

Therefore, inequality (23) holds for  $k + 1$ , concluding the proof.  $\square$

**Remark 1.** *It is worth mentioning that if the error sequence  $\{\theta_k\}$  given in step 0 of Newton-CondG method converges to zero, then it follows from (21) that the sequence  $\{x_k\}$  generated by Newton-CondG method is superlinear convergent to  $x^*$ . Also, under assumption **h3**, it follows from (22) that if  $\lambda = 0$ , then  $\{\|x_{k+1} - x^*\|/\|x_k - x^*\|^{p+1}\}$  is bounded, improving the superlinear convergence of  $\{x_k\}$ . However, we shall point out that  $\lambda = 0$  implies that  $\theta_k = 0$  for all  $k$ , which in turn may impose a stringent stopping criterium for CondG procedure (see step **P2** with  $\varepsilon = 0$ ). In the next section, we consider two classes of nonlinear functions whose majorant functions satisfy **h3**.*

## 4 Convergence results under Hölder-like and Smale conditions

In this section, we specialize Theorem 6 for two classes of functions. In the first one,  $F'$  satisfies a Hölder-like condition [5, 12], and in the second one,  $F$  is an analytic function satisfying a Smale condition [23, 24].

**Theorem 7.** *Let  $\kappa = \kappa(\Omega, \infty)$  as defined in (11). Assume that there exist a constant  $K > 0$  and  $0 < p \leq 1$  such that*

$$\|F'(x_*)^{-1}(F'(x) - F'(x_* + \tau(x - x_*)))\| \leq K(1 - \tau^p)\|x - x_*\|^p, \quad x \in B(x_*, \kappa), \quad \tau \in [0, 1]. \quad (27)$$

Take  $\lambda \in [0, 1)$  and let

$$\bar{r} := \min \left\{ \kappa, \left[ \frac{(1 - \lambda)(p + 1)}{K(2p + 1 - \lambda)} \right]^{1/p} \right\}.$$

*If  $\{\theta_k\} \subset [0, \lambda^2/2]$  and  $x_0 \in C \cap B(x_*, \bar{r}) \setminus \{x_*\}$ , then Newton-CondG method generates a sequence  $\{x_k\}$  which is contained in  $B(x_*, \bar{r}) \cap C$ , converges to  $x_*$  and satisfies*

$$\|x_{k+1} - x_*\| \leq \frac{(1 + \lambda)pK}{(p + 1)[1 - K\|x_0 - x_*\|^p]} \|x_k - x_*\|^{p+1} + \lambda\|x_k - x_*\|, \quad \forall k \geq 0.$$

Moreover, if  $t_0 = \|x_0 - x_*\|$ , then there holds

$$\|x_{k+1} - x_*\| \leq t_{k+1} := \frac{(1 + \lambda)pK t_k^{p+1}}{(p + 1)[1 - K t_k^p]} + \lambda, \quad \forall k \geq 0.$$

*Proof.* It is easy to prove that  $f : [0, \infty) \rightarrow \mathbb{R}$  defined by  $f(t) = Kt^{p+1}/(p + 1) - t$  is a majorant function for  $F$  on  $B(x_*, \kappa)$ , which satisfies **h3** in Theorem 6. Moreover, in this case, it is easily seen that  $\nu$  and  $\rho$ , as defined in (2) and (4), respectively, satisfy

$$\rho = \left[ \frac{(1 - \lambda)(p + 1)}{K(2p + 1 - \lambda)} \right]^{1/p} < \nu = [1/K]^{1/p},$$

and, as a consequence,  $\bar{r} = \min\{\kappa, \rho\} = r$  (see (20)). Therefore, the statements of the theorem follow from Theorem 6.  $\square$

**Remark 2.** As already mentioned, if the function  $F$  has Lipschitz continuous derivative then (27) is satisfied with  $p = 1$ , and then Theorem 7 holds for such a class of nonlinear functions. Additionally to the assumptions of Theorem 7, if  $\{\theta_k\}$  converges to zero, then  $\{x_k\}$  converges superlinear to  $x^*$ . Moreover, if  $\lambda = 0$  (i.e.,  $\theta_k = 0$  for all  $k$ ), we obtain the quadratic convergence rate of  $\{x_k\}$ .

In the previous theorem, we analyzed convergence of Newton-CondG method under Hölder-like condition. Next, we present a similar result for the class of analytic functions satisfying a Smale condition.

**Theorem 8.** Assume that  $F : \Omega \rightarrow \mathbb{R}^n$  is an analytic function and

$$\gamma := \sup_{n \geq 1} \left\| \frac{F'(x_*)^{-1} F^{(n)}(x_*)}{n!} \right\|^{1/(n-1)} < +\infty. \quad (28)$$

Let  $\lambda \in [0, 1]$  be given and compute

$$\bar{r} := \min \left\{ \kappa, \frac{5 - 3\lambda - \sqrt{(5 - 3\lambda)^2 - 8(1 - \lambda)^2}}{4(1 - \lambda)\gamma} \right\},$$

where  $\kappa = \kappa(\Omega, 1/\gamma)$  is as defined in (11). If  $\{\theta_k\} \subset [0, \lambda^2/2]$  and  $x_0 \in C \cap B(x_*, \bar{r}) \setminus \{x_*\}$ , then Newton-CondG method generates a sequence  $\{x_k\}$  which is contained in  $B(x_*, \bar{r}) \cap C$ , converges to  $x_*$  and satisfies

$$\|x_{k+1} - x_*\| \leq \frac{\gamma}{2(1 - \gamma\|x_0 - x_*\|)^2 - 1} \|x_k - x_*\|^2 + \sqrt{2\theta_k} \|x_k - x_*\|, \quad \forall k \geq 0.$$

Moreover, if  $t_0 = \|x_0 - x^*\|$ , then there holds

$$\|x_{k+1} - x_*\| \leq t_{k+1} := \frac{(1 + \lambda)\gamma t_k^2}{2(1 - \gamma t_k)^2 - 1} + \sqrt{2\theta_k} \lambda, \quad \forall k \geq 0.$$

*Proof.* Under the assumptions of the theorem, the real function  $f : [0, 1/\gamma) \rightarrow \mathbb{R}$  defined by  $f(t) = t/(1 - \gamma t) - 2t$  is a majorant function for  $F$  on  $B(x_*, 1/\gamma)$ , see for instance, [6, Theorem 14]. Since  $f'$  is convex, it satisfies **h3** in Theorem 6 with  $p=1$ , see [6, Proposition 7]. Moreover, in this case, it is easily seen that  $\nu$  and  $\rho$ , as defined in (2) and (4), respectively, satisfy

$$\rho = \frac{5 - 3\lambda - \sqrt{(5 - 3\lambda)^2 - 8(1 - \lambda)^2}}{4(1 - \lambda)\gamma}, \quad \nu = \frac{\sqrt{2} - 1}{\sqrt{2}\gamma}, \quad \rho < \nu < \frac{1}{\gamma},$$

and, as a consequence,  $\bar{r} = \min\{\kappa, \rho\} = r$  (see (20)). Therefore, the statements of the theorem follow from Theorem 6.  $\square$

## 5 Numerical experiments

This section summarizes the results of the numerical experiments we carried out in order to verify the effectiveness of Newton-CondG method. In the following experiments, we considered 25 box-constrained nonlinear systems, i.e., problem (1) with  $C = \{x \in \mathbb{R}^n : l \leq x \leq u\}$ , where  $l, u \in \mathbb{R}^n$ . We analyze the set of 25 problems specified in Table 1. These well-known problems come from different applications and some of them are considered challenging ones.

Table 1: The box-constrained nonlinear systems considered

Problem	Name and source	n
Pb 1	Himmelblau function [3, 14.1.1]	2
Pb 2	Equilibrium Combustion [3, 14.1.2]	5
Pb 3	Bullard-Biegler system [3, 14.1.3]	2
Pb 4	Ferraris-Tronconi system [3, 14.1.4]	2
Pb 5	Brown's almost linear system [3, 14.1.5]	5
Pb 6	Robot kinematics problem [3, 14.1.6]	8
Pb 7	Circuit design problem [3, 14.1.7]	9
Pb 8	Series of CSTRs $R = 0.935$ [3, 14.1.8]	2
Pb 9	Series of CSTRs $R = 0.940$ [3, 14.1.8]	2
Pb 10	Series of CSTRs $R = 0.945$ [3, 14.1.8]	2
Pb 11	Series of CSTRs $R = 0.950$ [3, 14.1.8]	2
Pb 12	Series of CSTRs $R = 0.955$ [3, 14.1.8]	2
Pb 13	Series of CSTRs $R = 0.960$ [3, 14.1.8]	2
Pb 14	Series of CSTRs $R = 0.965$ [3, 14.1.8]	2
Pb 15	Series of CSTRs $R = 0.970$ [3, 14.1.8]	2
Pb 16	Series of CSTRs $R = 0.975$ [3, 14.1.8]	2
Pb 17	Series of CSTRs $R = 0.980$ [3, 14.1.8]	2
Pb 18	Series of CSTRs $R = 0.985$ [3, 14.1.8]	2
Pb 19	Series of CSTRs $R = 0.990$ [3, 14.1.8]	2
Pb 20	Series of CSTRs $R = 0.995$ [3, 14.1.8]	2
Pb 21	Chemical reaction problem [16, Problem 5]	67
Pb 22	A Mildly-Nonlinear BVP [16, Problem 7]	451
Pb 23	H-equation, $c = 0.99$ [21, Problem 4]	100
Pb 24	H-equation, $c = 0.9999$ [21, Problem 4]	100
Pb 25	A Two-bar Framework [16, Problem 1]	5

The computational results were obtained using MATLAB R2015a on a 2.5 GHz intel Core i5 with 4GB of RAM and OS X system. In our implementation, the Jacobian matrices were approximated by finite differences and the error parameter  $\theta_k$  was set equal to  $10^{-5}$  for all  $k$ . Moreover, CondG Procedure stopped when either the required accuracy was obtained or the maximum of 300 iterations were performed. In order to compare Newton-CondG with other methods, we decided to keep the stopping criteria  $\|F(x_k)\|_\infty \leq 10^{-6}$  and a failure was declared if the number

of iterations was greater than 300. Furthermore, we also tested the method with initial points  $x_0 = l + 0.25\gamma(u - l)$  with  $\gamma = 1, 2, 3$ . However, since the choice  $\gamma = 3$  corresponds to an initial point that is a solution of Pb5 and the Jacobian matrices of Pb6 and Pb22 are singular at the initial point obtained with  $\gamma = 2$ , we used  $\gamma = 2.5$  in these cases.

Table 2 shows the performance of Newton-CondG method for solving 23 of the 25 problems considered. The other two problems (Pb 3 and Pb 7) do not appear in Table 2, because the method was not able to solve them for none of the three choices of initial points. In the table, “ $\gamma$ ” and “Iter” are the constant  $\gamma$  used to compute initial point  $x_0$  and the number of iterations of Newton-CondG method, respectively, “Time” is the CPU time in seconds and “ $\|F\|_\infty$ ” is the infinity norm of  $F$  at the final iterate  $x_k$ . Finally, the symbol “\*” indicates a failure.

From Table 2, we see that our method successfully ended 63 times on a total of 75 runs which shows its robustness. For comparison purposes, let us consider the performance of three methods analyzed in [1], namely, Scaled Trust-Region Newton (STRN), Active Set-Type Newton (ASTN) and Inexact Gauss-Newton-Type (IGNT) methods introduced in [1, 14, 16], respectively. Analyzing the numerical results in our Table 2 and the ones in Tables 2, 3, 4 and 5 in [1, Section 4] for the 24 common problems, the numbers of success on a total of 72 runs are 60, 58, 55 and 65, for Newton-CondG, STRN, ASTN and IGNT methods, respectively. These results indicate that Newton-CondG method is as effective as the other methods aforementioned for the set of problems considered. Moreover, it is worth to point out that the numbers of the function and Jacobian evaluations of Newton-CondG method are equal to number of iterations where as usual we do not take into account the functions evaluations due to finite-difference approximations of the Jacobians. However, for STRN, ASTN and IGNT methods the number of the function evaluations are, in general, greater than the number of iterations. This lower cost evaluations may reflect in computational savings. Therefore, we may conclude the applicability and effectiveness of our method.

## Conclusion

In this paper, we considered the problem of solving a constrained system of nonlinear equations. We proposed and analyzed a method which consists of a combination of Newton and conditional gradient methods. The convergence analysis was done via the majorant conditions technique, which allowed us to prove convergence results for different families of nonlinear functions. Under reasonable assumptions, we were able to provide a convergence radius of the method and establish some convergence rate results. In order to show the performance of our method, we carried out some numerical experiments and comparisons with some other methods were presented. It would be interesting for future research to combine the conditional gradient method with other Newton-like methods.

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Table 2: Performance of Newton-CondG method for 23 problems described in Table 1

Problem	$\gamma$	Iter	Time	$\ F\ _\infty$	Problem	$\gamma$	Iter	Time	$\ F\ _\infty$
1	1	6	5.5e-2	2.7e-8	15	1	5	3.6e-2	1.5e-7
	2	4	1.5e-2	1.2e-9		2	7	1.1e-2	4.6e-12
	3	4	1.3e-2	1.0e-7		3	9	1.2e-2	9.1e-10
2	1	11	2.7e-2	1.0e-7	16	1	8	1.7e-2	5.0e-7
	2	13	3.0e-2	4.7e-7		2	6	9.7e-3	1.2e-12
	3	14	3.3e-2	8.3e-7		3	9	1.1e-2	4.6e-11
4	1	4	6.6e-2	1.2e-7	17	1	3	1.1e-2	1.9e-7
	2	5	1.6e-2	1.2e-9		2	5	1.3e-2	7.3e-10
	3	5	1.7e-2	4.5e-13		3	9	7.4e-3	4.8e-13
5	1	10	2.9e-2	4.2e-8	18	1	3	3.1e-2	1.8e-8
	2	*				2	5	9.7e-3	3.2e-14
	2.5	*				3	8	1.0e-2	2.1e-8
6	1	5	1.4e-1	6.3e-8	19	1	3	3.9e-2	7.9e-10
	2.5	5	2.2e-2	1.3e-11		2	51	4.1e-2	7.9e-10
	3	5	1.0e-1	1.0e-11		3	45	4.0e-2	7.9e-10
8	1	17	3.3e-2	1.5e-8	20	1	3	2.8e-2	4.9e-12
	2	28	1.8e-1	1.5e-8		2	6	9.4e-3	5.1e-12
	3	10	1.9e-2	6.3e-12		3	11	1.4e-2	5.1e-12
9	1	90	1.0e-1	2.0e-7	21	1	20	4.6e+0	6.5e-7
	2	*				2	*		
	3	10	1.5e-2	1.9e-12		3	*		
10	1	*			22	1	14	1.3e+1	1.1e-7
	2	7	1.4e-2	6.9e-8		2.5	16	1.5e+1	1.9e-8
	3	9	1.7e-2	7.8e-7		3	20	1.9e+1	1.3e-10
11	1	14	4.2e-2	1.5e-10	23	1	5	4.5e-1	4.0e-12
	2	7	9.7e-3	7.5e-12		2	6	5.2e-1	3.8e-9
	3	9	1.5e-2	3.4e-7		3	6	5.3e-1	2.5e-10
12	1	24	4.5e-2	5.9e-12	24	1	7	6.0e-1	1.4e-7
	2	6	9.3e-3	6.0e-9		2	9	7.4e-1	6.7e-9
	3	9	1.1e-2	1.2e-7		3	7	6.0e-1	4.8e-8
13	1	7	2.9e-2	1.2e-8	25	1	18	1.5e-1	4.3e-7
	2	6	9.5e-3	1.5e-7		2	20	1.0e-1	7.9e-7
	3	9	1.3e-2	3.7e-8		3	21	4.5e-2	6.8e-7
14	1	5	4.3e-2	6.9e-10					
	2	8	1.7e-2	6.0e-10					
	3	9	1.1e-2	7.6e-9					